## Spatial Interpolation

- Spatial Interpolation (SI) is a process of using points with known values to estimate values at the other points.
- SI are grouped into *global* and *local* methods.
- A **global method** uses every control point available in estimating the unknown values.
- A **local method** uses a sample of control points for estimation.

### Control points

- Control points are points with known values.
- A basic assumption of SI:
  - *The value to be estimated at a point is more influenced by nearby control points than those that are farther away.*
- Control points should be well distributed in the study area, but this rarely occur in the real world (see Figure 13.1) and causes problems in spatial interpolation.
Global Methods

Trend Surface Analysis

- Approximate points with known values with a polynomial equation, the trend surface model is then used to estimate values at other points. (Figure 13.2)
- A linear or first-order trend surface equation:

\[ z_{x,y} = b_0 + b_1 x + b_2 y \]

- \( Z \) = attribute values, is a function of \( x \) and \( y \)
- \( b \) = coefficients, are estimated from the control points.
Goodness of fit can be measured by coefficient of determination $r^2$.

Natural phenomena are complex and require higher-order trend surface, e.g., a cubic or third-order trend surface, based on the equation:

$$z_{x,y} = b_0 + b_1 x + b_2 y + b_3 x^2 + b_4 xy + b_5 y^2 + b_6 x^3 + b_7 x^2 y + b_8 xy^2 + b_9 y^3$$

- See Box 13.2 and Figure 13.3
Box 13.2 A Worked Example of Trend Surface Analysis

Figure 13.2 shows five weather stations with known values around point 0 with an unknown value. The table below shows the x-, y-coordinates of the points, measured in row and column of a grid with the cell size of 2000 meters, and their known values.

<table>
<thead>
<tr>
<th>Row</th>
<th>Col</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>69</td>
<td>76</td>
</tr>
<tr>
<td>2</td>
<td>79</td>
<td>89</td>
</tr>
<tr>
<td>3</td>
<td>85</td>
<td>92</td>
</tr>
<tr>
<td>4</td>
<td>58</td>
<td>65</td>
</tr>
<tr>
<td>5</td>
<td>73</td>
<td>88</td>
</tr>
<tr>
<td>0</td>
<td>69</td>
<td>72</td>
</tr>
</tbody>
</table>

This example shows how we can use Equation 13.1, or a linear trend surface, to interpolate the unknown value at point 0. The least-squares method is commonly used to solve for the coefficients of $b_0, b_1,$ and $b_2$ in Equation 13.1. Therefore, the first step is to set up three normal equations, similar to those for a regression analysis.

The equations can be rewritten in matrix form as

$$
\begin{bmatrix}
\begin{array}{c}
1 & x_0 & y_0 \\
1 & x_1 & y_1 \\
1 & x_2 & y_2 \\
1 & x_3 & y_3 \\
1 & x_4 & y_4 \\
1 & x_5 & y_5 \\
1 & x_0 & y_0 \\
1 & x_1 & y_1 \\
1 & x_2 & y_2 \\
1 & x_3 & y_3 \\
1 & x_4 & y_4 \\
1 & x_5 & y_5 \\
\end{array}
\end{bmatrix}
\begin{bmatrix}
b_0 \\
b_1 \\
b_2 \\
\end{bmatrix}
=
\begin{bmatrix}
Z_0 \\
Z_1 \\
Z_2 \\
Z_3 \\
Z_4 \\
Z_5 \\
\end{bmatrix}
$$

Using the values of the five known points, we can calculate the statistics and substitute the statistics into the equation.

We can then solve the $b$ coefficients by multiplying the inverse of the first matrix on the left by the matrix on the right.

Using the coefficients, the unknown value at point 0 can be estimated by

$$
Z_0 = b_0 + b_1 x_0 + b_2 y_0
$$

Figure 13.3
An isoline map of a third-order trend surface created from 105 control points with annual precipitation values.
Regression Models

- Relates a dependent variable to a number of independent variables in an equation, which can then be used for prediction or estimation.
- Variables can be non-spatial or spatial attributes.

Local Methods

Thiessen polygons (Voronoi polygons)

- Polygons constructed around the known points so that any point within the Thiessen polygon is closer to the polygon’s known point than any other known points.
- The procedure starts with connecting known points to form triangles, usually use the Delaunay triangulation. (Figure 13.4)
- Then connecting lines perpendicular to the sides of each triangle at their midpoints.
Density Estimation

- Measures densities in a grid based on a distribution of points and their known values.
- Estimate by placing a grid on a point distribution, count the points that fall within each cell, divide the total point value by the cell size.
- See Figure 13.5.

Figure 13.4
The diagram shows control points, Delaunay triangulation in dashed lines, and Thiessen polygons in thin solid lines.
Inverse Distance Weighting (IDW)

- IDW is a local method.
- Assumptions:
  - ‘The unknown value of a point is influenced more by the nearby control points than those far away.’
- The degree of influence, or the weight, is expressed by the inverse of the distance between points raised to a power.
- The power of 1.0 means constant rate of change. The power of 2.0 suggests that the rate of change in values is higher near a known point and level off away from it.
The general equation for the IDW method is:

\[ z_0 = \frac{\sum_{i=1}^{s} Z_i \cdot \frac{1}{d_j^k}}{\sum_{i=1}^{s} \frac{1}{d_j^k}} \]

Where:
- \( z_0 \) = estimated value at point 0
- \( z_i \) = value at control point \( i \)
- \( d_j \) = distance between control point \( i \) and point 0
- \( s \) = the number of control points used in estimation
- \( k \) = specified power.
See Figure 13.8,13.9 and Box 13.5 for details.

**Thin plate Spline (TPS)**

- **TPS** create a surface that passes through control points and has the least possible change in slope at all points.
- TPS fit the control points with a minimum-curvature surface.
The approximation of thin-plate spline is:

\[ Q(x, y) = \sum A_i d_i^2 \log d_i + a + bx + cy \]

Where \( x, y \) = coordinates of the points to be interpolated.

\[ d_i^2 = (x - x_i)^2 + (y - y_i)^2 \]

\( x_i, y_i \) = coordinates of control point \( i \)

\((a + bx + cy)\) = local trend function
Where \( n \) = number of control points
\[ f_i = \text{known value at control point } i \]
- The estimation of coefficients requires \( n+3 \) simultaneous equations.
- Other algorithms may be used (See Chang’s p. 253-254).
- Thin-plate spline and their variations are recommended for smooth, continuous surfaces such as elevation, and water table.
- See also Box 13.6, Figure 13.10, 13.11

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**Thin-plate spline with tension**

\[
a + \sum_{i=1}^{n} A_i R(d_i)
\]

Where \( a = \text{trend function, and the basis function } R(d) \) is:

\[
-\frac{1}{2\pi\phi^2} \ln \left( \ln \left( \frac{d\phi}{2} \right) + c + K_\phi(d(\phi)) \right)
\]

Where \( \phi = \text{weight used with the tension method, if } \phi \text{ is close to 0, approximation is similar to basic TPS. The higher } \phi \text{ reduces the stiffness of the plate.} \)
This example uses the same data set as in Box 13.2 but interpolates the unknown value at point 0 by the splines with tension method. The method first involves calculation of $R(d)$ in Equation 13.15 using the distances between the point to be estimated and the control points, distances between the control points, and the $\delta$ value of 0.1. The following table shows the $R(d)$ values along with the distance values.

<table>
<thead>
<tr>
<th>Point</th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$d_3$</th>
<th>$d_4$</th>
<th>$d_5$</th>
<th>$d_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>3.23</td>
<td>3.45</td>
<td>2.34</td>
<td>4.56</td>
<td>2.34</td>
<td>3.45</td>
</tr>
<tr>
<td>A2</td>
<td>2.34</td>
<td>3.45</td>
<td>4.56</td>
<td>2.34</td>
<td>3.45</td>
<td>4.56</td>
</tr>
<tr>
<td>A3</td>
<td>4.56</td>
<td>2.34</td>
<td>3.45</td>
<td>4.56</td>
<td>2.34</td>
<td>3.45</td>
</tr>
</tbody>
</table>

The matrix solutions are:

$$a = 13.203 \quad A_1 = 0.396 \quad A_2 = -0.226$$
$$A_3 = -0.058 \quad A_4 = -0.047 \quad A_5 = -0.065$$

Now we can calculate the value at point 0 by

$$P_0 = 13.203 + (0.396)(-7.510) + (-0.226)(-9.879) + (-0.058)(16.831) + (-0.047)(-18.574) + (-0.065)(-22.834) = 15.795$$

The splines with tension method is one of the thin-plate splines methods discussed in the text. Using the same data set, the estimation of $P_0$ by other methods is as follows: 16.350 by thin plate splines and 15.015 by regularized splines (using the $\delta$ value of 0.1).

Figure 13.10
An isolist map created by the regularized splines method.

Figure 13.11
An isolist map created by the splines with tension method.