Numerical Simulation of Blood Flow in the Stenosed Coronary Artery

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Abstract

This study focuses on the behavior of three dimensional blood flows in a stenosed coronary artery. A numerical model based on the finite element method is developed to simulate the blood flow through the arterial domain, including the lumen region and the porous wall. The fluid flow in the lumen region is governed by the continuity equation and the Navier-Stokes equations, while blood flow in the porous wall is described by the Brinkman equations. The velocity field, the pressure and the wall shear stresses in the lumen and porous wall are computed in a fully coupled manner through the use of the lumen-wall interface condition. The work shows that the results obtained from the porous wall model are significantly different from those obtained by the rigid-wall model.

KEY WORDS: Blood flow, stenotic artery, Brinkman model, Carreau model, finite element method.

Introduction

Cardiovascular disease is one of the major causes of deaths in developed countries. Most cases are associated with some form of abnormal flow of blood in stenotic arteries. In recent years, surgical treatments of cardiovascular disease have been developed rapidly, and coronary artery bypass grafting (CABG) has been widely used for patients with very severe stenosis. A large number of bypass grafts are implanted worldwide each year. However, up to 25 percents of grafts fail in one year and up to 50 percents fail in ten years after surgery. To achieve a higher success rate of bypass operations, it is essential to have information of the rheological behavior of blood, and to predict blood flow, pressure distribution and wall shear stress in a stenotic artery.

Over the last decade, researchers have focused on the study of wall shear stress in blood vessels because of its central role in the maintenance of vascular tone, vascular remodeling, and localization of atherosclerosis (DePaola et al., 1992; Sill et al., 1995). Lipowsky (1995) found that the mean value of the wall shear stress on endothelial cells was in the order of 10 dyn/cm² in arteries and might be higher in capillaries and lower in post capillary venules. It has also been established that high shear stresses on the wall are correlated with various degree of stenotic artery (Holme et al., 1997; Marano et al., 1998).

Comparison to all the validated velocity points along the vessel's lumen indicates that the results of the model in a rigid pipe are not directly related to the in-vivo experiment data. In addition, a little has been done to investigate the blood flow in the lumen and the porous arterial wall using the fluid-wall condition at the interface (Tada and Tarbell, 2000; Stangeby and Ethier, 2002; Tada and Tarbell, 2004). Therefore, further development of mathematical models to study blood flow is necessary. Once a satisfactory model is generated, the future management of human health by mathematical modeling will be possible.

In this study, a three-dimensional numerical model based on the finite element method is developed to simulate the blood flow patterns in the lumen and the porous wall. The luminal blood transport is governed by the transient convection-diffusion equations and the transport in the porous wall is governed by the Brinkman model. The fluid flows in the lumen and the porous wall are computed in a fully coupled manner through the use of the lumen/wall condition. The effect of
stenosis size on flow pattern and wall shear stress is investigated. The results obtained from the vessel models having the porous wall are then compared with those obtained from rigid wall model.

Mathematical model

Figure 1 shows a typical cross section of the arterial porous wall consisting of three embedded layers including the tunica intima, the tunica media and the adventitia. The innermost layer is the tunica intima which consists of a thin layer of endothelial cells, connective tissue and basement membrane. The middle layer is the tunica media which comprises the smooth muscle cells and a continuous interstitial fluid phase of proteoglycan and collagen fiber. The outermost layer is the adventitia which is made up mostly of stiff collagenous fibers having an elastic modulus of $10^8$-$10^9$ dyn/cm². Blood is transported mainly in the artery lumen but some could be transported through the wall layers including the tunica initima and the tunica media.

![Figure 1: A cross section structure of the arterial porous vessel.](image)

In this work the flow through the lumen and the porous wall, including the lunica intimal and lunica media, is considered. The vessel wall is modeled as a porous asymmetric membrane and to keep the model simple, the deformation of blood vessels is neglected in this study.

Various non-Newtonian models have been proposed to describe the constitutive relation of blood, including the Power law model, the Carreau model and etc. In this work, we use the Carreau model in which the stresses are related to the deformation rate by

\[
\sigma_{ij} = -p_1 I + 2\mu_n (\dot{\gamma}) D_{ij} \quad \text{[1]}
\]

where the viscosity \( \mu_n \) is given by

\[
\mu_n = \mu_\infty + (\mu_0 - \mu_\infty)\left[1 + (\lambda \dot{\gamma})^2\right]^{(n-1)/2}
\]

in which \( \mu_0, \mu_\infty, \lambda \) and \( n \) are constants, \( p_1 \) denotes pressure in the lumen region.

The computational domain consists of two regions: the porous arterial wall \( \Omega_W \) and the lumen \( \Omega_L \). The velocity fields in the lumen and in the porous arterial wall are computed in a fully coupled manner through the use of the lumen/wall condition at the interface \( \partial \Omega_{L/W} \).

In the luminal region, \( \Omega_L \), the equations governing the blood flow include the constitutive equation [1] and the following continuity equation and the stress equations of motion,

\[
u_{L,i} = 0, \quad \text{[2]}
\]

\[
\rho \left( \frac{\partial u_{i}}{\partial t} + u_{j} u_{i,j} \right) = \frac{\partial \sigma_{ji}}{\partial x_j} + F_{w, i}, \quad \text{[3]}
\]
where \( \rho \) denotes the blood density which is \( 1.06 \text{ g cm}^{-3} \), \( u_i \) represents the component of velocity vector in the \( i \)th direction, and \( F_i \) is the volume force acting on the fluid. By substituting [1] into [3], we have the following Navier-Stokes equations

\[
\rho \left( \frac{\partial u_i}{\partial t} + u_i u_{i,j} \right) = -p_{1,j} + \left( \mu_n (u_{i,j} + u_{j,i}) \right)_j + F_{1i},
\]

in the porous wall, \( \Omega_w \), blood flow is described by the following continuity equation and the Brinkman equations,

\[
v_{i,j} = 0, \quad \rho \frac{\partial v_i}{\partial t} + \frac{\mu}{\kappa} v_i = -p_{2,j} + \left( \mu (v_{i,j} + v_{j,i}) \right)_j + F_{2i},
\]

where \( \mu \) denotes viscosity in porous layer, \( \kappa \) is permeability, \( v_i \) represents the component of the velocity vector in the \( i \)th direction, \( p_2 \) denotes pressure in the wall, and \( F_2 \) is the volume force acting on the fluid in the wall.

The boundary conditions considered for the velocity field and pressure field include the Dirichlet type and the Neumann/Robin type, i.e., for \( i,j = 1, 2, 3 \)

\[
\begin{align*}
u_i &= \bar{u}_i(t) = \frac{Q(t)}{A}, u_2 = u_3 = 0 \quad \text{on } \partial \Omega_{in} \quad \text{[7]} \\
p_1 &= p_0(t), \left( \mu_n (u_{i,j} + u_{j,i}) \right)_\mathbf{n} = 0 \quad \text{on } \partial \Omega_{out}
\end{align*}
\]

where \( A \) denotes the inlet cross section area of an artery, \( Q(t) \) and \( p_0(t) \) are the pulsatile flow rate and the pulsatile pressure which have been expressed as Fourier series in our previous work (Amornsamankul et al., 2006).

On the interface between the lumen and the porous wall, the expression for the pressure and velocity must be continuous across the interface. We thus set

\[
v_{|L/w} = u_{|L/w} \cdot \quad \text{[8]}
\]

On other wall surfaces, we apply the no-slip condition.

**Method of solution**

Based on the model presented in the previous section, the boundary value problem for the problem in question is as follows.

Find \( p_1, u_1, u_2, u_3 \) and \( p_2, v_1, v_2, v_3 \in H^1_{\Omega} \) such that for all test functions \( \tilde{u}_i \in H^1_{0u} (\Omega) \), \( \tilde{v}_j \in H^1_{0v} (\Omega) \), and \( \tilde{p}_1, \tilde{p}_2 \in H^1 (\Omega) \), all the Dirichlet boundary conditions for the unknown functions are satisfied and

\[
\begin{align*}
\nabla \cdot (\mathbf{u}, \tilde{p}_1) &= 0, \quad \text{[9]} \\
\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \nabla \mathbf{u}, \tilde{\mathbf{u}} \right) - \nabla \cdot (\mu_n \left( \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right) \tilde{\mathbf{u}}) + (\nabla p_1, \tilde{\mathbf{u}}) &= (\mathbf{f}, \tilde{\mathbf{u}}), \quad \text{[10]} \\
\nabla \cdot (\mathbf{v}, \tilde{p}_2) &= 0, \quad \text{[11]} \\
\rho \left( \frac{\partial \mathbf{v}}{\partial t} - \nabla \cdot (\mu \nabla \mathbf{v} + (\nabla \mathbf{v})^T) \tilde{\mathbf{v}} \right) + \frac{\mu}{\kappa} (\mathbf{v}, \tilde{\mathbf{v}}) + (\nabla p_2, \tilde{\mathbf{v}}) &= (\mathbf{g}, \tilde{\mathbf{v}}), \quad \text{[12]}
\end{align*}
\]

where \( (\cdot, \cdot) \) denotes the inner product on the square integrable function space \( L^2 (\Omega), H^1 (\Omega) \) is the
Sobolev space $W^{1,2}(\Omega)$ with norm $\|\cdot\|_{2,\Omega}, H^1_0(\Omega) = \{ v \in H^1(\Omega) | v = 0 \text{ on the boundary} \}$. A standard procedure is then carried out to reduce the second-order derivatives involved in the above problem into the first-order ones using integration by parts to ensure that all integrals involved are well defined. By using the Galerkin finite element formulation, we obtain the following system of ordinary differential equations,

$$\mathbf{MU} + \mathbf{KU} = \mathbf{F}, \quad \text{[13]}$$

where $\mathbf{U} = [v_f \ v_B \ v_w]$ in which $v_f$ and $v_w$ denote quantities in the lumen region and the porous wall region, and $v_B$ represents quantities on the interface of the lumen region and the porous wall region, the matrix $\mathbf{M}$ corresponds to the transient terms, the matrix $\mathbf{K}$ corresponds to the advection and diffusion terms, $\mathbf{F}$ represents the forcing function.

To solve the above system, a standard backward Euler scheme has been used to determine the velocity, pressure and wall shear stress at any instant of time.

### Numerical results and discussion

The examples under consideration are stenotic arteries having porous wall with 25 %, 50 % and 65 %-area severity. The computational domain is a straight tube with length 5 cm and the diameter of the lumen is 0.21 cm. The thickness of the wall is 0.05 cm. There is a stenosis with spherical curvature in the middle part on one side of the internal wall as shown in Figure 2.

![Figure 2: The 3-D geometry of the 50 % stenotic tubes with finite element mesh.](image)

The three-dimensional geometry tubes having 25 %, 50 % and 65 %-area severity are respectively discretized into 10,685 tetrahedron elements with 67,504 degrees of freedom (velocity and pressure), 10,624 tetrahedron elements with 67,133 degrees of freedom, and 12,983 tetrahedron elements with 77,616 degrees of freedom. The solutions were computed for 5 cardiac cycles to ensure reproducibility of the pulsed characteristic flow.

Figure 3 (a-b) shows the distribution of velocity $u_e$ in three different luminal cross sections of the 65 % stenotic tube at the peak of systole $t=3.15$s: (a) at an upstream cross section $x=2.0$cm, and (b) at the downstream cross section $x=2.7$cm from the inlet.

![Figure 3: The velocity field $u_e$ in the lumen region of the 65 % stenotic tube at the peak of systole $t=3.15$s: (a) at $x=2$cm, and (b) at $x=2.7$cm.](image)

Figure 4 (a-b) show the velocity field $u_e$ on three wall cross sections of the 65 % stenotic tube at the peak of systole $t=3.15$s: (a) at an upstream cross section $x=2.0$cm, and (b) at the downstream cross
section \( x=2.7 \text{cm} \) from the inlet. When the blood passes through the narrowing channel with 65 \%-area severity, jet flow is present at downstream and yields the high shear stresses on the vessel wall which is in the same side of the stenosis base. This can damage the endothelial cells and block the blood flow into the wall. The results show that in the downstream region, there is not enough blood supplying to the porous wall.

To study the effect of wall-interaction on the flow field and pressure field, we compare the results obtained from the models having the porous wall and the model with a rigid wall for the cases of 25 \%, 50 \% and 65 \%-area severity at the peak of systole \( t=3.15 \text{s} \). Results for the cases of 25 \% and 65 \%-area severity are shown in Figures 5 (a-b). The pressure along a longitudinal line and the velocity along the throat line are investigated. It is found that the fluid-wall interaction has significant effect on the flow field and the pressure field. The velocity field and the pressure field in the model having porous wall are lower than the ones obtained from the model with a rigid wall during a cardiac cycle for the case of 25%-area severity, which is different to the case of 65%-area severity.

Figure 4: The velocity field \( u_x \) in the porous wall of the 65 \% stenotic tube at the peak of systole \( t=3.15 \text{s} \) : (a) at \( x=2 \text{cm} \), and (b) at \( x=2.7 \text{cm} \).

![Figure 4](image)

Figure 5: Pressure along the longitudinal line and the throat-line mean flow of the stenosis in the tube at the peak of systole having (a) 25\%, and (b) 65 \%-area severity of stenosis.

![Figure 5](image)

In a healthy artery, the wall shear stress is approximately 15 dyne/cm\(^2\) (Glagov et al., 1988). The interior surface will be damaged, once the wall shear stress reaches a value higher than 400 dyne/cm\(^2\) (Ku, 1997). Therefore to determine the critical flow condition, prediction of wall shear stress using numerical experiments becomes necessary.

To capture the effect of wall-interaction on the wall shear stresses, we investigate the models with 65 \%-area severity at the peak of systole. The solutions are plotted on the plane representing the wall surface where the stenosis is located at the center. The results show that in the model with the porous wall, at the peak of systole the wall shear stress around the stenosis site varies between -965 and -200 dyne/cm\(^2\) as shown in Figure 6 (a). In the model with the solid wall, at the peak of systole the wall shear stress around the stenosis site varies between -1117 and -300 dyne/cm\(^2\). At the downstream from the stenosis, the wall shear stress varies between -200 and -100 dyne/cm\(^2\) as shown in Figure 6 (b). We may conclude that wall shear stresses in the model having the porous wall are lower than the ones in the model having the solid wall during a cardiac cycle.
Conclusions

The convective-diffusive transport of blood in the luminal channel and in the arterial porous wall has been studied numerically using a three-dimensional mathematical model and a numerical technique based on the finite element method. The results obtained from the vessel models having 25%, 50% and 65%-area severity show that the pressure drops very quickly around the stenosis site and creates a jet flow at the throat of the stenosis and that a larger percent-area severity of stenosis leads to a higher pressure jump, higher blood speeds around the stenosis site and higher wall shear stresses.

To analyze the critical flow in the stenotic artery, we focus on the model with 65%-area severity. It is found that the critical flow occurs around the stenosis site and the downstream from the stenosis. When the blood passes through the narrowing channel with 65%-area severity, jet flow is present at downstream and yields the high shear stresses on the vessel wall which is in the same side of the stenosis base. This can damage the endothelial cells and block the blood flow into the wall. The results show that in the downstream region, there is not enough blood supplying to the porous wall. Comparing the results obtained from the models with the porous wall and the model with a rigid wall, it is found that the fluid-wall interaction has significant effect on the flow pattern, the pressure distribution and the wall shear stress. Velocity field, pressure field and wall shear stresses obtained from the model having porous wall are found to be lower than the ones obtained from the model having a rigid wall during a cardiac cycle.

References


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